University of California, Riverside
Department of Mathematics

Mathematics 10B - Calculus of Several Variables
SECTION 030 - Dr. Henriques

Sample Midterm Questions

SPRING 2011

1. Consider the double integral $\int_0^2 \int_y^2 e^{x^2}\, dx dy$

   (a) Sketch the region that is determined by the limits of integration.

   (b) Evaluate the integral by reversing the order of integration.

2. Evaluate the integral
$$\int_0^2 \int_{x^2}^{3x} x + y \, dy \, dx$$
and sketch the region that is determined by the limits of integration.

3. Use double integrals to evaluate the area bounded by the curves $y = 3x$, $x = 0$, $y = 1 - (x-1)^2$ and sketch the corresponding region.

4. Let W be the region in $\mathbb{R}^3$ bounded by the spheres $x^2 + y^2 + z^2 = 1$, $z = 0$ and $z = \sqrt{3}/2$. Determine the value of
$$\iiint_W \frac{1}{\sqrt{x^2 + y^2 + z^2}} \, dV$$

5. Let W be the region in $\mathbb{R}^3$ bounded by the spheres $x^2 + y^2 = 3z^2$, $z = 0$ and $z = -3$. Determine the value of
$$\iiint_W \frac{3x + y}{\sqrt{x^2 + y^2}} \, dV$$

6. Let W be the region in $\mathbb{R}^3$ bounded by the two spheres $x^2 + y^2 + z^2 = 16$, $x^2 + y^2 + z^2 = 25$ and $z = 0$ for which $z \geq 0$. Determine the value of
$$\iiint_W \frac{1}{\sqrt{x^2 + y^2 + z^2}} \, dV$$

7. Let W be the region in $\mathbb{R}^3$ bounded by the two spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$. Determine the value of
$$\iiint_W \frac{1}{2} \, dV$$

8. Let W be the region in $\mathbb{R}^3$ with $z \geq 0$ that is bounded by the three surfaces $z = 6/\sqrt{2}$, $z^2 = 36 - x^2 - y^2$ and $z = 0$. Determine the value of
$$\iiint_W \frac{3}{x^2 + y^2 + z^2} \, dV$$

9. Let W be the region in $\mathbb{R}^3$ with $z \geq 0$ that is bounded by the xy-plane and the three dimensional sphere of radius 2 and center at the origin. Determine the value of

$$\iiint_W \frac{e^{x^2+y^2+z^2}}{x^2+y^2+z^2}\, dV$$

10. Let D be the triangular region bounded by the coordinate axes and the line $x + y = 1$. Compute

$$\iint_D \cos(x+y)\, dA.$$

11. Transform the integral given in Cartesian coordinates into spherical coordinates and evaluate the new integral:

$$\int_0^5 \int_{-\sqrt{25-z^2}}^{\sqrt{25-z^2}} \int_{-\sqrt{25-y^2-z^2}}^{\sqrt{25-y^2-z^2}} dx\, dy\, dz\,.$$

12. Transform the integral given in Cartesian coordinates into spherical coordinates and evaluate the new integral:

$$\int_{\sqrt{3}}^2 \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} \int_{-\sqrt{4-y^2-z^2}}^{0} dx\, dy\, dz\,.$$

13. Let W be the region in the first octant bounded by $y^2 + z^2 = 9$ and the planes $y = x$, $x = 0$, and $z = 0$. Use cylindrical coordinates to evaluate the integral:

$$\iiint_W z\, dV\,.$$

14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T(u,v) = (\,x(u,v)\,,\, y(u,v)\,)$ where $\begin{cases} x(u,v) = \frac{1}{2}v + u \\ y(u,v) = \frac{1}{2}v \end{cases}$.

   (a) Express u and v as functions of x, y:

   (b) Sketch the region D^* of the uv-plane that is transformed into the closed rectangle $D = [0,1] \times [0,1]$ in the xy-plane.

   (c) Use the change of variables theorem to rewrite the following integral in the coordinates u and v:

$$\int_0^1 \int_0^1 4y\, e^{x-y}\, dx\, dy$$

15. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(u,v,w) = (\,x(u,v,w)\,,\, y(u,v,w)\,,\, z(u,v,w)\,)$ where $\begin{cases} x(u,v) = 3u + w \\ y(u,v) = v + 2w \\ z(u,v) = w - v \end{cases}$.

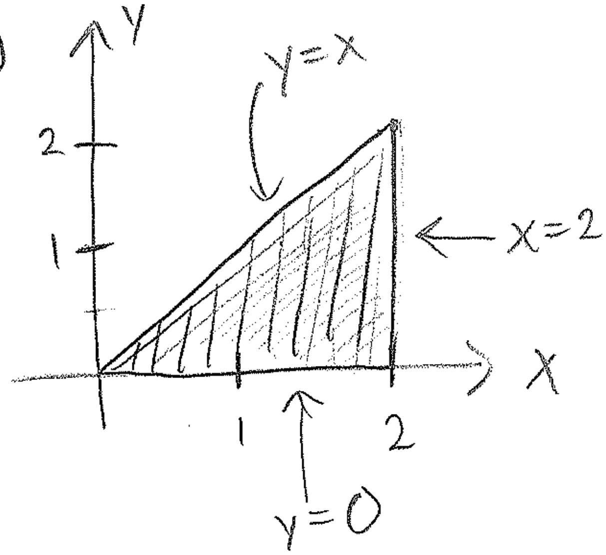
   (a) Express u, v and w as functions of x, y and z:

(b) Sketch the region D^* of the uvw-space that is transformed into the closed rectangle $D = [0,1] \times [0,1] \times [0,1]$ in the xyz-space.

(c) Use the change of variables theorem to rewrite the following integral in the coordinates u, v and w:

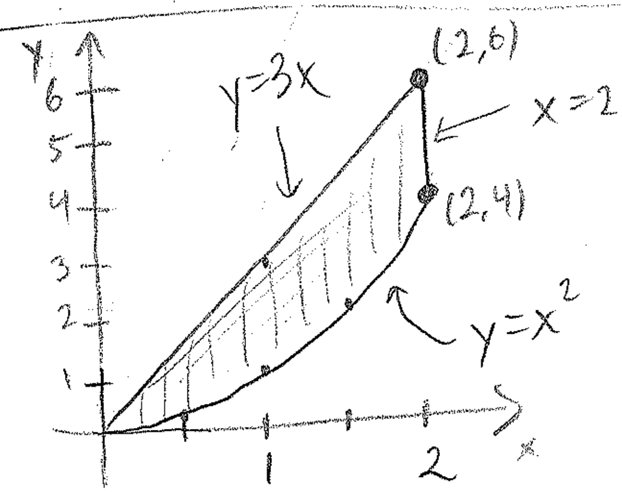
$$\int_0^1 \int_0^1 \int_0^1 e^{x-y+z}\, dx\, dy\, dz$$

1 (a)



(b) $\int_0^2 \int_y^2 e^{x^2}\,dx\,dy = \int_0^2 \int_0^x e^{x^2}\,dy\,dx$

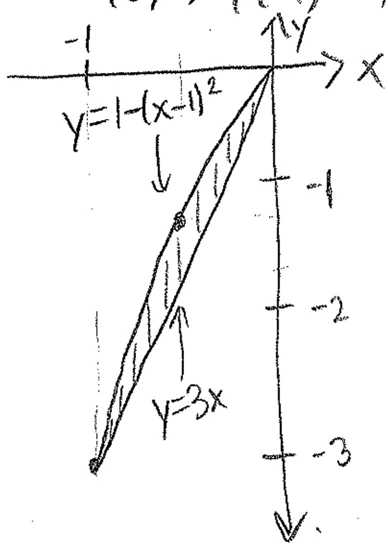
2



(not to scale)

3 Intersection: $3x = 1-(x-1)^2 = 1 - x^2 + 2x - 1 \Rightarrow x^2 + x = 0 \Rightarrow x = 0, -1$

(0,0) & (-1,-3)

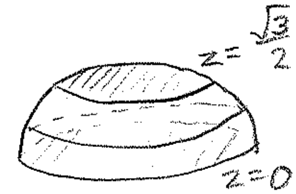


(sketch)

$$\text{Area} = \int_{-1}^{0} \int_{3x}^{1-(x-1)^2} dy\,dx$$

④ $$\iiint_W \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV = \int_{\frac{\pi}{6}}^{\pi}\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{\rho^2}}\rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$z = \frac{\sqrt{3}}{2} \Rightarrow z = \rho\cos\varphi = \frac{\sqrt{3}}{2}$ if $\rho = 1 \Rightarrow \varphi = \frac{\pi}{6}$.
(on the sphere)



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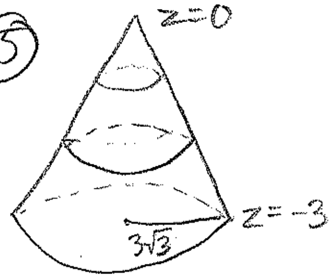
⑤



Cylindrical: $x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$$\iiint_W \frac{3x+y}{\sqrt{x^2+y^2}}\,dV = \int_0^{3\sqrt{3}}\int_0^{2\pi}\int_{-3}^{\frac{-r}{\sqrt{3}}} \frac{3r\cos\theta + r\sin\theta}{\sqrt{r^2}}\, r\, dz\, d\theta\, dr$$

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⑥ $$\iiint_W \frac{1}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_4^5 \frac{1}{\sqrt{\rho^2}}\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

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⑦ This can be done 2 ways:

(i) $$\iiint_W \frac{1}{2}\,dV = \int_0^{\pi}\int_0^{2\pi}\int_1^2 \frac{1}{2}\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi.$$

(ii) $S_1 : x^2+y^2+z^2 \le 1$, $S_2 : x^2+y^2+z^2 \le 4$. Volume of sphere $V = \frac{4}{3}\pi r^3$

$$\iiint_W \frac{1}{2}\,dV = \frac{1}{2}\iiint_W dV = \frac{1}{2}\text{ Volume of } W = \frac{1}{2}\left(\text{Volume of } S_2 - \text{Volume of } S_1\right)$$

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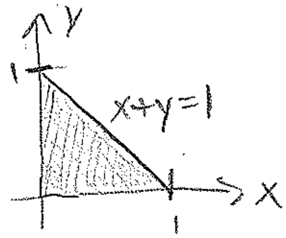
⑧ $z = \frac{6}{\sqrt{2}} \Rightarrow z = \rho\cos\varphi = \frac{6}{\sqrt{2}}$ if $\rho = 6 \Rightarrow z = 6\cos\varphi = \frac{6}{\sqrt{2}} \Rightarrow \cos\varphi = \frac{1}{\sqrt{2}} \Rightarrow \varphi = \frac{\pi}{4}$
(on sphere)

$$\iiint_W \frac{3}{x^2+y^2+z^2}\,dV = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^6 \frac{3}{\rho^2}\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

(9) $$\iiint_W \frac{e^{x^2+y^2+z^2}}{x^2+y^2+z^2}\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^{2} \frac{e^{\rho^2}}{\rho^2}\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

(10)



$$\iint_D \cos(x+y)\,dA = \int_0^1\int_0^{1-x}\cos(x+y)\,dy\,dx \overset{\substack{u=x+y\\ du=dy}}{=} \int_0^1\int_x^1 \cos u\,du\,dx$$

(11) $$\int_0^5\int_{-\sqrt{25-z^2}}^{\sqrt{25-z^2}}\int_{-\sqrt{25-y^2-z^2}}^{\sqrt{25-y^2-z^2}} dx\,dy\,dz = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{5} \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

(12) $$\int_{\sqrt{3}}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{-\sqrt{4-y^2-z^2}}^{0} dx\,dy\,dz = \int_0^{\frac{\pi}{6}}\int_{\pi}^{2\pi}\int_0^{2} \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

(13) $$\iiint_W z\,dV = \int_0^3\int_0^{\frac{\pi}{2}}\int_0^{r\cos\theta} r\sin\theta\, r\,dx\,d\theta\,dr$$

$$\boxed{y=r\cos\theta,\ z=r\sin\theta,\ x=x}$$

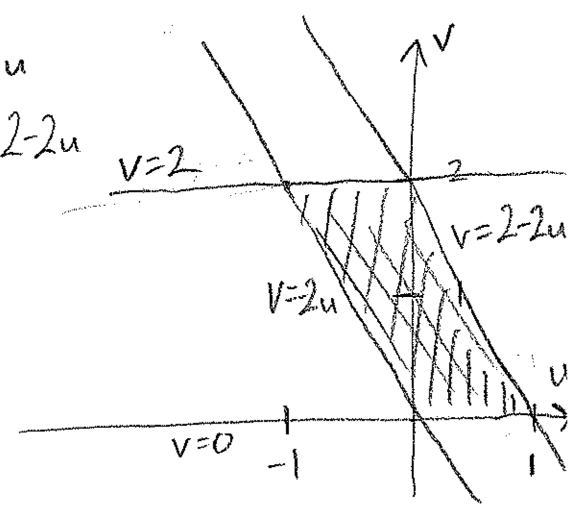
(14) (a) $v=2y$, $u=x-y$

(b) $x=0 \longrightarrow u=-y \;\&\; v=2y \longrightarrow u=-\frac{1}{2}v \rightarrow v=-2u$

$x=1 \longrightarrow u=1-y \;\&\; v=2y \longrightarrow u=1-\frac{1}{2}v \rightarrow v=2-2u$

$y=0 \longrightarrow v=0$

$y=1 \longrightarrow v=2$



(c) $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$

$$\int_0^1 \int_0^1 4ye^{x-y}\,dx\,dy = \int_0^2 \int_{-\frac{v}{2}}^{1-\frac{v}{2}} 2ve^{u}\left(\frac{1}{2}du\,dv\right)$$

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(15) (a) $\begin{cases} 3u \quad\quad + w = x & ① \\ \quad v + 2w = y & ② \\ \; -v + w = z & ③ \end{cases}$

②+③ : $3w = y+z \Rightarrow \boxed{w = \frac{y+z}{3}}$

$w \mapsto$ ① : $3u + \frac{y+z}{3} = x \Rightarrow \boxed{u = \frac{3x-y-z}{9}}$

③: $v = w - z = \frac{y+z}{3} - z = \boxed{\frac{y-2z}{3} = v}$